Chapter 1
Computing bounds of the MTTF for a set of Markov Chains

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Abstract We present an algorithm to find some upper and lower bounds of the Mean Time To Failure (MTTF) for a set of absorbing Discrete Time Markov Chains (DTMC). We first present a link between the MTTF of an absorbing chain and the steady-state distribution of an ergodic DTMC derived from the absorbing one. The proposed algorithm is based on the polyhedral theory developed by Courtois and Semal and on a new iterative algorithm which gives bounds of the steady-state distribution of the associated ergodic DTMC at each iteration.

1.1 Introduction

Finite DTMC models provide a very efficient technique for the study of dynamical systems. However, in many engineering problems, it is still hard to give the precise parameters to describe the chain: we need the exact transition probabilities between the states of the chain. In many cases, we only know an interval for these transition probabilities. This is equivalent to state that the underlying model M is inside of a set of chains described by an entry-wise lower bounding matrix **L** and an entry-wise upper bounding matrix **U**. Many results have been proposed to find bounds on the steady-state distribution when the chain is ergodic (see for instance the polyhedral theory developed by Courtois and Semal [4] and applied by Muntz and his colleagues [7] for reliability problem and more recently by Buchholz [2] or ourselves

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for a faster algorithm [1]). In [8], the authors give the algorithms for the efficient model checking of such models.

Here we investigate a new problem. We assume that the chain is absorbing and the transition probabilities are imprecise (partially known). More formally, we assume that the chain belongs to a set of absorbing Markov chains having the same set of absorbing states, $\mathscr{A}$. We will denote by $\preceq$ the element-wise comparison of two vectors (or matrices). We also assume that the matrix of the chain $\mathbf{M}$ satisfies $\mathbf{L} \preceq \mathbf{M} \preceq \mathbf{U}$. We study how to find bounds for the MTTF of a such model $\mathbf{M}$. We show first the relation between the MTTF of an absorbing chain and an associated ergodic DTMC built from the absorbing one. We then use the polyhedral theory and the same arguments to construct bounds on MTTF. We use the new numerical technique given in [3] to solve the steady-state distribution of each matrix considered in the polyhedral approach we have already proposed in [1] to study imprecise DTMCs. This algorithm that we apply after some on imprecise absorbing provides at each iteration upper and lower bounds of the MTTF. Therefore, we obtain bounds at the first iteration and at each iteration the bounds are improved. Our algorithm is also numerically stable as it is only based on the product of non negative vectors and matrices. In the following of the paper, we describe how to link the MTTF of an absorbing DTMC with the steady-state distribution of an ergodic associated DTMC. Then in section 3, we briefly introduce the $I\nabla L$ and $I\nabla U$ Algorithms [3]. In section 4, we present how we can combine all these results to derive a bound for the MTTF for a set of Markov chains. We illustrate the approach with some numerical results.

1.2 Mean Time To Failure

Let us first begin with some notations. All vectors are row vectors, $\mathbf{e}_i$ is a row vector with 1 in position i and 0 elsewhere, the vector with all entries equal to 0 is denoted by $\mathbf{0}$ and $\mathbf{Id}$ is used for the identity matrix. Finally, x^t denotes the transposed vector of x and $||x||$ is the sum of the elements of vector x.

We consider a model defined by an absorbing DTMC noted by $\mathbf{M}$. We assume that we have several absorbing states and no recurrent class. We want to compute the mean time to reach an absorbing state. We show how to transform this initial problem to the construction of an ergodic DTMC and the analysis of its steady-state distribution. More formally, we suppose that we use the following matrix decomposition: $\mathbf{M} = \left[\begin{array}{c|c} \mathbf{Id} & \mathbf{0} \\ \hline \mathbf{R} & \mathbf{Q} \end{array}\right]$, once we have organised the state space to have the absorbing states (i.e. in set $\mathscr{A}$) before the transient states (i.e. in set $\mathscr{T}$).

The absorbing DTMCs are studied through their fundamental matrices [9]. By assuming that there is no recurrent class, we have the following well-known results:

- The fundamental matrix $\mathbf{F} = (\mathbf{Id} - \mathbf{Q})^{-1}$ exists.
- We assume that there exist several absorbing points. The probability to be absorbed in state j knowing that the initial state is i is equal to $(\mathbf{F}.\mathbf{R})[i, j]$.
- The mean time before absorption knowing that initial state is i is $(\mathbf{F}.e^t)[i]$.

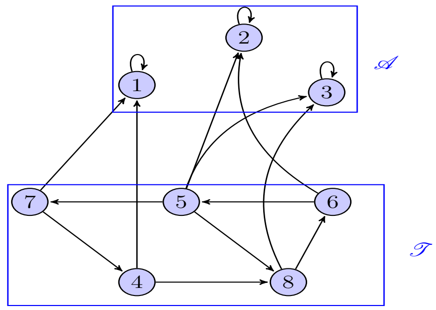

Fig. 1.1 Transition graph of an absorbing Markov Chain.

We propose to compute the mean absorbing time, called also MTTF through the steady-state probabilities of an ergodic DTMC built by the underlying absorbing one. We assume that the directed graph of the transient states $\mathscr{T}$ is strongly connected. Note that it does not imply that there is a recurrent class among these states.

Let i be an arbitrary non absorbing state. We consider a new matrix built as follows. First, we aggregate all the absorbing states into one state which is the first one of the state space. Thus, matrix $\mathbf{R}$ is also summed up into a vector r^t. Second, we add a loop on state 1 with probability 0.5. Third, we modify the first row: we add a vector denoted as p_i whose entries are all equal to zero except entry i which is 0.5. Finally, the built stochastic matrix, $\mathbf{M}_i$ is as follows:

$$\mathbf{M}_i = \left[\begin{array}{c|c} 1/2 & p_i \\ \hline r^t & \mathbf{Q} \end{array}\right]. \tag{1.1}$$

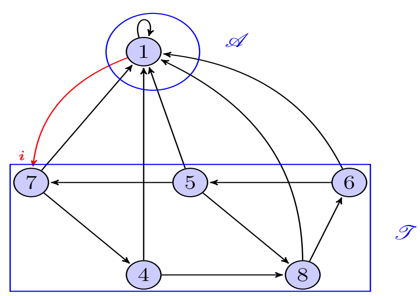

Fig. 1.2 Transition graph of the new Markov Chain to obtain the time before being absorbed knowing that we begin at state 7.

In the following, set $\mathscr{F}$ will denote the state space of $\mathbf{M}_i$. It contains set $\mathscr{T}$ and one state which represents the aggregation of the absorbing states of $\mathbf{M}$.

Property 1 *Matrix* $\mathbf{M}_i$ *is ergodic. Therefore the steady-state distribution of* $\mathbf{M}_i$ *(denoted as* π_i*) exists.*

Proof. The chain is finite. The graph of the states in $\mathscr{T}$ is strongly connected and there is a directed edge between state 1 and state i and between a state in $\mathscr{T}$ and state 1 because vector r is not zero. Therefore, the graph on the whole state space is strongly connected. Finally, as there is a loop at state 1, the chain is aperiodic. Thus, the chain is ergodic and the steady-state distribution exists.

The question is to find a relation between the MTTF of $\mathbf{M}$ and π_i computed from matrix $\mathbf{M}_i$. Let $E[T_i]$ be the mean time before the absorption knowing the initial state is i for the absorbing chain $\mathbf{M}$. On the other hand, for the ergodic DTMC $\mathbf{M}_i$, we know that $1/\pi_i[1]$ is the mean time between two visits to state 1. To compute it, we condition on the first transition out of state 1. We have two possible transitions: a loop in state 1 with probability 1/2, so the time between two visits is 1 and a transition into state i with probability 1/2, so the time between two visits is $(1+E[T_i])$. Therefore:

$$\frac{1}{\pi_i[1]} = \frac{(1+E[T_i])}{2} + \frac{1}{2}.$$

Finally, we obtain $E[T_i]$:

$$E[T_i] = \frac{2}{\pi_i[1]} - 2. \tag{1.2}$$

We have to find bounds on π_i when the matrix is specified by matrices $\mathbf{L}$ and $\mathbf{U}$. To do this, we combine Muntz's approach for imprecise DTMCs and the iterative algorithm presented in [1]. The bounds on $\pi_i[1]$ will then provide bounds on MTTF by using Eq. 1.2.

Let us first begin with the algorithm for exact calculation of $\pi_i[1]$ thus $E[T_i]$ before proceeding with imprecise Markov chains.

1.3 Algorithms based on monotone sequences

Let $\mathbf{P}$ be a finite stochastic matrix. We assume that $\mathbf{P}$ is ergodic. We first introduce some quantities easily computed from $\mathbf{P}$.

Definition 1. Set $\nabla_P[j] = \mathbf{min}_i \mathbf{P}[i,\, j]$ and $\triangle_P[j] = \mathbf{max}_i \mathbf{P}[i,\, j]$. Remark that ∇_P may equal to vector $\mathbf{0}$ but Δ_P is positive as the chain is irreducible.

Bušić and Fourneau [3] proposed two iterative algorithms based on simple (**max**, +)(resp. (**min**,+)) properties, called $I\nabla L$ (resp. $I\nabla U$) which provide at each iteration a new lower (resp. upper) bound $x^{(k)}$ (resp. $y^{(k)}$) of the steady state distribution of $\mathbf{P}$.

Theorem 1. *Let* $\mathbf{P}$ *be an irreducible and aperiodic stochastic matrix with steady state probability distribution* π*. If* $\nabla_P \neq \mathbf{0}$*, Algorithm* $I\nabla L$ *provides at each iteration lower bounds for all components of* π *and converges to* π *for any value of the parameters a and b such that* $a \preceq \pi$*,* $b \preceq \nabla_P$ *and* $b \neq \mathbf{0}$*.*

Algorithm 1 Algorithm Iterate ∇ Lower Bound (I∇L)

Require: $a \preceq \pi$, $b \preceq \nabla_{\mathbf{P}}$ and $b \neq \mathbf{0}$.
Ensure: Successive values of $x^{(k)}$.

1: $x^{(0)} = a$.
2: **repeat**
3: $\quad x^{(k+1)} = \mathbf{max}\left\{x^{(k)}, x^{(k)}\mathbf{P} + b(1 - ||x^{(k)}||)\right\}$.
4: **until** $1 - ||x^{(k)}|| < \varepsilon$.

One can check that the conditions on the initialisation part of the algorithm require that $||\nabla_{\mathbf{P}}|| > 0$. Similarly, we have proved another algorithm (called $I\nabla U$) to compute a decreasing sequence $y^{(k)}$ of iterative upper bounds. It is based on an initialization with vector $\triangle_P$ and an iteration with operator **min** instead of **max**. Note that combining both theorems we obtain a proved envelope for all the components of vector π. It is also proved in [3] that the norm of the envelope converges to zero faster than a geometric with rate $(1 - ||b||)$. The algorithms have been implemented in a tool called XBorne [5]. These algorithms also have two important properties. First, under some technical conditions, an entry-wise bound on the stochastic matrices provides an entry-wise bound on the steady-state distribution (see [3]). Second, they deal with infinite matrix with some constraints on the associated directed graph [6].

Example 1. Let $\mathbf{P}$ be a stochastic matrix $\mathbf{P} = \begin{pmatrix} 0.6 & 0 & 0.2 & 0.2 & 0 \\ 0.4 & 0.2 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0.2 & 0.3 & 0.2 \\ 0.2 & 0 & 0.2 & 0.3 & 0.3 \\ 0.1 & 0 & 0 & 0.4 & 0.5 \end{pmatrix}$.
We have $\nabla_{\mathbf{P}} = (0.1, 0, 0, 0.2, 0)$. For $\varepsilon = 10^{-5}$, algorithm I∇L with $a = b = \nabla_{\mathbf{P}}$ provides the following sequence of lower bounds for the probabilities.

k	1	2	3	4	5	$1 - \|\|x^{(k)}\|\|$
1	0.17	0	0.06	0.22	0.06	0.4900
3	0.2413	0.0102	0.1092	0.2546	0.1446	0.2401
7	0.2869	0.0169	0.1409	0.2826	0.2151	0.0576
11	0.2968	0.0183	0.1481	0.2897	0.2332	0.0139
21	0.2997	0.0188	0.1502	0.2920	0.2389	0.0004
31	0.2997	0.0188	0.1503	0.2921	0.2391	$1.1\ 10^{-5}$

1.4 Bounds of the MTTF

We now present how to deal with imprecise Markov chains. Muntz and his coauthors [7] have proposed an approach for bounding steady-state availability. The theoretical background is based on Courtois and Semal polyhedral results on steady-state distribution [4]. We only present here a weak form of the theorem.

Theorem 2. *Given a lower bound* $\mathbf{L}$ *of the transition probability matrix of a given DTMC (let's assume that the chain has n states), we can compute a bounds for its steady-state probability vector* π. *In a first step one compute the steady-state solution of n DTMCs. Transition probability matrix* $\mathbf{L}^s$ *associated with the* s^{th} *DTMC is obtained from sub-stochastic matrix* $\mathbf{L}$ *by increasing the elements of column s to make* $\mathbf{L}^s$ *stochastic. Let* π^s *be the steady state probability vector solution of the* s^{th} *DTMC. The lower (resp. upper) bound on the steady state probability of state* j *is computed as* $\mathbf{min}_s \pi^s[j]$ *(resp.* $\mathbf{max}_s \pi^s[j]$*).*

$$\mathbf{min}_s\, \pi^s[j] \leq \pi[j] \leq \mathbf{max}_s\, \pi^s[j]. \tag{1.3}$$

We have showed in [1] how one can combine theorem 2 and $I\nabla L$ and $I\nabla U$ algorithms to prove new algorithms which provide at each iteration a new component-wise bounds on steady state distribution. To simplify the presentation, we only present the upper bound case (for the lower bound see [1]). The main idea behind the upper bounding algorithm is to compute first, for all $s \in \mathscr{F}$ an upper bound $Y^{(k),s}$ associated with matrix $\mathbf{L}^s$ with $I\nabla U$ algorithm. Then, we apply Muntz's result to deduce an upper bound on steady state distribution of π. This process is iterated until the stopping criterion is reached. The sequences $Y^{(k),s}$ converges faster than a geometric with rate $(1-||\nabla_{\mathbf{L}^s}||)$. Once all these sequence have converged, the **max** operator between distributions proved by the polyhedral theory does not change either.

Algorithm 2 Algorithm Iterate ∇ Bounds for imprecise Markov chains

Require: $\forall s \in \mathscr{F},\ \alpha[s] > 0$.
Ensure: Successive values of $Y^{(k)}$ and $X^{(k)}$.

1: $\forall s \in \mathscr{F}, \mathbf{L}^s = \mathbf{L} + \alpha^t e_s, c^s = \triangle_{\mathbf{L}^s}, b^s = \nabla_{\mathbf{L}^s}, Y^{(0),s} = c^s, X^{(0),s} = b^s$.
2: **repeat**
3: $\quad \forall\, s \in \mathscr{F}, Y^{(k+1),s} = \mathbf{min}\left\{Y^{(k),s}, Y^{(k),s}\mathbf{L}^s + b^s(1-||Y^{(k),s}||)\right\}$.
4: $\quad Y^{(k+1)} = \mathbf{max}_s \left\{Y^{(k+1),s}\right\}$.
5: $\quad \forall\, s \in \mathscr{F}, X^{(k+1),s} = \mathbf{max}\left\{X^{(k),s}, X^{(k),s}\mathbf{L}^s + b^s(1-||X^{(k),s}||)\right\}$.
6: $\quad X^{(k+1)} = \mathbf{min}_s \left\{X^{(k+1),s}\right\}$.
7: **until** $\sum_s(||Y^{(k),s}||-1) < \varepsilon$ and $\sum_s(1-||X^{(k),s}||) < \varepsilon$.

Theorem 3. *Let* $\mathbf{L}$ *be an irreducible sub-stochastic matrix, algorithm 2 provides at each iteration k an element wise upper and lower bounds on the steady-state distribution of any ergodic matrix entry-wise larger than* $\mathbf{L}$.

For a proof and some arguments on complexity, see [1]. We combine all these results to obtain bounds on the MTTF. We consider an absorbing non-negative matrix $\mathbf{M}$ define by $\mathbf{L} \preceq \mathbf{M} \preceq \mathbf{U}$, such that each row sum is less than or equal to 1.

First, we consider a set of stochastic matrices $\mathscr{P} = \{\mathbf{L}^s \,|\, \mathbf{L}^s$ is an irreducible stochastic matrix and $\mathbf{L}^s \succeq \mathbf{L}\}$. s is a column index. $\mathbf{L}^s$ is matrix $\mathbf{L}$ where elements in the sth column have been increased as necessary to make the matrix stochastic. We first add the following assumption: **in the following we assume that** $r(k) > 0$ **for all** k**.** Therefore $\nabla(\mathbf{L}^s) > 0$ and we can apply the algorithms. For each matrix $\mathbf{M} \in \mathscr{P}$, we define transition matrix $\mathbf{M}_{i,s}$ as in Equation 1.1. We use Muntz's approach and the results in [1] about the previous algorithms. Finally, the steady-state distribution using the Algorithm 2 at iteration n is bounded by:

$$\mathbf{min}_s\{X^{(n),s}\} \leq \pi_i \leq \mathbf{max}_s\{Y^{(n),s}\}.$$

Then, the bounds for the average time $E[T_i]$ are:

$$E[\underline{T_i}] = \frac{2}{\mathbf{max}_s\{Y^{(n),s}\}[1]} - 2 \leq E[T_i] \leq \frac{2}{\mathbf{min}_s\{X^{(n),s}\}[1]} - 2 = E[\overline{T_i}].$$

And we conclude with the theorem which states the result and we give a small example.

Theorem 4. *Let* $\mathbf{L}$ *be a sub-stochastic matrix, Algorithm 2 provides at each iteration* k *a lower bound and an upper bound ot the MTTF of any absorbing matrix entry-wise larger than* $\mathbf{L}$ *and defined on the same set of transient states* $\mathscr{T}$.

Example 2. Consider absorbing matrix

$$\mathbf{L} = \left(\begin{array}{rr|rrrr} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0.30 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 \\ 0.20 & 0.3 & 0.2 & 0.1 & 0 & 0.1 \\ 0.25 & 0.2 & 0.1 & 0.2 & 0.2 & 0 \\ 0.10 & 0 & 0.2 & 0.1 & 0.2 & 0.1 \end{array}\right).$$

Assume that we want to compute the expected time before being absorbed when we begin at state $i = 5$. First, we aggregate state 1 and 2, which are absorbing states in $\mathbf{L}$ and we modify the transition between the first state and state $i = 5$. Note that state 5 is now the fourth state due to aggregation of state 1 and 2.

We obtain an irreducible sub-stochastic matrix $\left(\begin{array}{r|rrrr} \boxed{0.5} & 0 & 0 & \boxed{0.5} & 0 \\ \hline 0.4 & 0.2 & 0.1 & 0.1 & 0.2 \\ 0.5 & 0.2 & 0.1 & 0 & 0.1 \\ 0.45 & 0.1 & 0.2 & 0.2 & 0 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.1 \end{array}\right)$.

Note that $r(k) > 0$ for all k. Thus, we can use the Nabla based algorithms as the first column of the matrix has all its entries positive. We then derive the five matrices which are in set $\mathscr{P}$:

$$M_{5,1} = \left(\begin{array}{c|cccc} 0.5 & 0 & 0 & 0.5 & 0 \\ \hline 0.4 & 0.2 & 0.1 & 0.1 & 0.2 \\ 0.6 & 0.2 & 0.1 & 0 & 0.1 \\ 0.5 & 0.1 & 0.2 & 0.2 & 0 \\ 0.4 & 0.2 & 0.1 & 0.2 & 0.1 \end{array}\right), M_{5,2} = \left(\begin{array}{c|cccc} 0.5 & 0 & 0 & 0.5 & 0 \\ \hline 0.4 & 0.2 & 0.1 & 0.1 & 0.2 \\ 0.5 & 0.3 & 0.1 & 0 & 0.1 \\ 0.45 & 0.15 & 0.2 & 0.2 & 0 \\ 0.1 & 0.5 & 0.1 & 0.2 & 0.1 \end{array}\right),$$

$$M_{5,3} = \left(\begin{array}{c|cccc} 0.5 & 0 & 0 & 0.5 & 0 \\ \hline 0.4 & 0.2 & 0.1 & 0.1 & 0.2 \\ 0.5 & 0.2 & 0.2 & 0 & 0.1 \\ 0.45 & 0.1 & 0.25 & 0.2 & 0 \\ 0.1 & 0.2 & 0.4 & 0.2 & 0.1 \end{array}\right), M_{5,4} = \left(\begin{array}{c|cccc} 0.5 & 0 & 0 & 0.5 & 0 \\ \hline 0.4 & 0.2 & 0.1 & 0.1 & 0.2 \\ 0.5 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0.45 & 0.1 & 0.2 & 0.25 & 0 \\ 0.1 & 0.2 & 0.1 & 0.5 & 0.1 \end{array}\right),$$

$$M_{5,5} = \left(\begin{array}{c|cccc} 0.5 & 0 & 0 & 0.5 & 0 \\ \hline 0.4 & 0.2 & 0.1 & 0.1 & 0.2 \\ 0.5 & 0.2 & 0.1 & 0 & 0.2 \\ 0.45 & 0.1 & 0.2 & 0.2 & 0.05 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.4 \end{array}\right).$$

The hybridation of Muntz and Nabla algorithms provides bounds at each iteration. At iteration $n = Diam^{\mathbf{L}} = 3$, we obtain the first non trivial lower bound on each component of the steady-state distribution. According to Table 2 the average time before being absorbed knowing that the initial state is state 5 is bounded by:

$$2.0072 \leq E[T_5] \leq 2.5043.$$

n	$E[\underline{T_5}]$	$E[\overline{T_5}]$	$\lvert\lvert\mathbf{max}_s\{Y^{(n),s}\}\rvert\rvert - 1$	$1 - \lvert\lvert\mathbf{min}_s\{X^{(n),s}\}\rvert\rvert$
3	1.8469	7.8571	0.4733	0.6623
4	1.9104	6.7711	0.3723	0.6009
13	2.0062	3.5435	0.1934	0.2671
34	2.0072	2.5982	0.1642	0.0910
64	2.0072	2.5078	0.1613	0.0745
84	2.0072	2.5043	0.1612	0.0739

1.5 Concluding remarks

We now plain to build a model checker for problems modeled with uncertain Markov chain and which will be based on the results in [1] and the algorithms we have presented here. Note that, we can also easily compute bounds on transient probabilities which are only based on the positivity of the matrices.

Proposition 1. *Let* $\mathbf{L} \preceq \mathbf{M}$ *and* $X_0^{\mathbf{L}}$ *a positive vector such that* $\|X_0^{\mathbf{L}}\| \leq 1$ *and* $X_0^{\mathbf{L}} \preceq \pi_0^{\mathbf{M}}$ *then, we have for all k,* $X_k^{\mathbf{L}} \preceq \pi_k^{\mathbf{M}}$ *where* $\pi_k^{\mathbf{M}}$ *is the distribution for the chain* $(\mathbf{M}, \pi_0^{\mathbf{M}})$ *and* $X_k^{\mathbf{L}} = X_0^{\mathbf{L}} \mathbf{L}^k$.

It is also possible to obtain lower bounds on the probability of being absorbed knowing a bound of the initial distribution.

Proposition 2. *Let* $\mathbf{L} \preceq \mathbf{M}$ *such that* $\mathbf{L}(i,i) = 1$, *for a state i. Let* $X_0^{\mathbf{L}}$ *be a positive vector such that* $\|X_0^{\mathbf{L}}\| \leq 1$ *and* $X_0^{\mathbf{L}} \preceq \pi_0^{\mathbf{M}}$. *Then for all k, the probability of being absorbed at state i knowing a lower bound of the initial distribution is lower bounded by* $(X_0 \mathbf{L}^k)(i)$ *and upper bounded by* $(X_0 \mathbf{L}^k)(i) + 1 - \sum_{j \neq \mathscr{B}} (X_0 \mathbf{L}^k)(j)$, *where* $\mathscr{B}$ *is the set of absorbing points of* $\mathbf{L}$ *(i.e.* $\mathbf{L}(j,j) = 1$*).*

Thus, we have obtained a set of algorithms which looks sufficient to address the numerical resolution of models based on uncertain discrete time Markov chains.

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